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A parametric spline method for second-order singularly perturbed boundary-value problem

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Abstract: *A numerical method based on parametric spline with adaptive parameter is given for the second-order singularly perturbed two-point boundary value problems of the form*
$\varepsilon y'' = p(x)y' + q(x)y + r(x);\ y(a) = \alpha_0;\ y(b) = \alpha_1$
The derived method is second-order and fourth-order convergence depending on the choice of the two parameters α and β. Error analysis of a method is briefly discussed. The method is tested on an example and the results found to be in agreement and support the predicted theory.
Keywords: *Singular perturbation; parametric spline functions; BVPs; ODEs.*

I. Introduction

Ordinary differential equations occur in many scientific disciplines, for instance in mechanics, chemistry, ecology, and economics. In addition, some methods in numerical partial differential equations convert the partial differential equation into an ordinary differential equation, which must then be solved. Some difference schemes for singularly perturbed differential equation are derived see [17] and [18]. In the paper we consider a second-order singularly perturbed boundary value problem

$$\varepsilon y'' = p(x)y' + q(x)y + r(x), \tag{1}$$

$$y(a) = \alpha_0, y(b) = \alpha_1 \tag{2}$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous, bounded, real functions and are parameters such that it is known that problem (1)-(2) exhibits boundary layers at one or both ends of the interval depending on the choice of the function $p(x)$ [6].

The problems in which a small parameter multiplies to a highest derivative arise in various yields of science and engineering, for instance fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, hydrodynamics etc, see [15] and the references therein. Fyfe [7] have developed Bickley [16] methods by considering the case of (regular) linear boundary-value problems. Our scheme for the corresponding problem (i.e. $\varepsilon = 1$, $p(x) = 0$) reduces to the Bickley scheme. However, it is well known since then that the cubic spline method of Bickley gives only $O(h^2)$ convergent approximations. But cubic spline itself is a fourth-order process [5]. Recently, Aziz [6] suggested a nonpolynomial parametric spline method based on the spline trigonometric basis $Span\{1,\ x,\ \cos kx,\ \sin kx\}$ where k is the frequency of the trigonometric part of the splines function. In the present paper, we apply nonpolynomial parametric spline functions that have a polynomial and hyper trigonometric part to develop a new numerical method for obtaining smooth approximations to the solution of such above boundary value problems (1) and (2). The new method is of order two for arbitrary a and b such that $\alpha + \beta = \frac{1}{2}$. Our method performs better than the other collocation, finite difference, and spline methods of same order and thus represents an improvement over existing methods (see Refs. [15],[20]).

The spline function we propose in this paper have the form $Span\{1,\ x,\ \cosh kx,\ \sinh kx\}$ where k is the frequency of the hyper trigonometric part of the splines function which can be real or pure imaginary and which will be used to raise the accuracy of the method. This approach has the advantage over finite difference methods [3] that it provides continuous approximations to not only for $y(x)$, but also for $y'(x)$, $y''(x)$ and higher derivatives at every point of the range of integration. Also the advantage of the method is that the coefficient matrix of the system is of the tridiagonal form, and the method has an order of convergence $O(h^4)$, where h is the step size. Section 2, we develop the new nonpolynomial parametric spline method for solving (1) and (2). The convergence analysis of the method is considered in Section 3. Section 4 contains numerical illustrations and results are compared with the method of Kadalbajoo and Bawa [10] to demonstrate the efficiency of the method.

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II. Derivation of the spline method

Let $x_0 = a, x_N = b, x_i = a + ih, h = (b - a)/N$.

A function $S(x,\mu)$ of class $C^2[a,\ b]$ which interpolates $y(x)$ at the mesh point depends on a parameter μ , and reduces to cubic spline in $[a,\ b]$ as $\mu \to 0$ is termed as parametric cubic spline function. The parametric spline function $S(x,\mu) = S_i(x,\mu)$ in $[x_{i-1}, x_i]$, satisfying the differential equation

$$s_i''(x) - \mu^2 s_i(x) = \left(M_{i-1} - \mu^2 y_{i-1}\right)\left(\frac{x_i - x}{h}\right) + \left(M_i - \mu^2 y_i\right)\left(\frac{x - x_{i-1}}{h}\right) \tag{3}$$

where $M_i = y''(x_i), s_i(x_i) = y(x_i)$ and μ is a parameter and we denote to $y(x_i)$ by y_i ,

Solving the differential equation (3) on the interval $[x_{i-1}, x_i]$, subject to $s_i(x_i) = y_i$ and $s_{i-1}(x_{i-1}) = y_{i-1}$ we obtain the parametric spline in the form

$$\begin{aligned} s_i(x) = {} & \frac{h^2}{k^2 \sinh k}\left[M_i \sinh\frac{k(x - x_{i-1})}{h} - M_{i-1}\sinh\frac{k(x_i - x)}{h}\right] \\ & - \frac{h^2}{k^2}\left[\left(M_i - \frac{k^2}{h^2}y_i\right)\frac{(x - x_{i-1})}{h} - \left(M_{i-1} - \frac{k^2}{h^2}y_{i-1}\right)\frac{(x_i - x)}{h})\right] \end{aligned} \tag{4}$$

where $k = \mu h$

Differentiating equation (4) and letting x tend to x_i , we obtain

$$S_i'(x_i) = \frac{hM_i \cosh k}{k^2 \sinh k} - \frac{hM_{i-1}}{k^2 \sinh k} + \frac{h}{k^2}(M_{i-1} - M_i) + \frac{y_i - y_{i-1}}{h} \tag{5}$$

Considering the interval $[x_i, x_{i+1}]$ the parametric spline function take the form

$$\begin{aligned} s_{i+1}(x) = {} & \frac{h^2}{k^2 \sinh k}\left[M_{i+1}\sinh\frac{k(x - x_i)}{h} - M_i \sinh\frac{k(x_{i+1} - x)}{h}\right] \\ & - \frac{h^2}{k^2}\left[\left(M_{i+1} - \frac{k^2}{h^2}y_{i+1}\right)\frac{(x - x_i)}{h} - \left(M_i - \frac{k^2}{h^2}y_i\right)\frac{(x_{i+1} - x)}{h})\right] \end{aligned} \tag{6}$$

and the first derivative at $x = x_i$ given by

$$S_{i+1}'(x_i) = \frac{hM_{i+1}}{k^2 \sinh k} - \frac{hM_i \cosh k}{k^2 \sinh k} + \frac{h}{k^2}(M_i - M_{i+1}) + \frac{y_{i+1} - y_i}{h} \tag{7}$$

The continuity of the first derivative of $S(x,\mu)$ at $\boldsymbol{x_i}$ in the form $s'(x_i) = s_{i+1}'(x_i)$. Equation (5) and (7) gives

$$y_{i+1} - 2y_i + y_{i-1} = h^2\left\{\alpha M_{i+1} - 2\beta M_i + \alpha M_{i-1}\right\} \tag{8}$$

where

$$\alpha = k^{-2}\left(1 - k\,\mathrm{csch}\,k\right) \tag{9}$$

$$\beta = k^{-2}\left(1 - k\coth k\right) \tag{10}$$

Now, from equation (1) substituting by $\varepsilon M_i = p(x_i)y'_i + q(x_i)y_i + r(x_i)$, in equation (8)
and using the following approximations for first derivative of *y*:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \tag{11}$$

$$y_{i+1}' = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} \tag{12}$$

$$y_{i-1}' = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} \tag{13}$$

We arrive at the following linear system which may be solved to get the approximations y_{i} of the solutions $y(x)$ at $\mathrm{x_i}, i = 1,2,3,\dots,N-1$ respectively:

$$\begin{aligned} & y_{i-1}\left(\frac{h}{2}\alpha p_{i+1} + h\beta p_i - \frac{3h\alpha}{2}p_{i-1} + h^2\alpha q_{i-1} - \varepsilon\right) + y_i\left(-2\alpha p_{i+1} - 2h^2\beta q_i + 2h\alpha p_{i-1} + 2\varepsilon\right) \\ & + y_{i+1}\left(\frac{3h}{2}\alpha p_{i+1} + h^2\alpha q_{i+1} - h\beta p_i - \frac{h}{2}\alpha p_{i-1} - \varepsilon\right) \\ & = -h^2(\alpha r_{i-1} - 2\beta r_i + \alpha r_{i+1}), \quad i = 1,2,3,\dots,N-1. \end{aligned} \tag{14}$$

When α=1/6,β=1/3 ; our method reduces to the Kadalbajoo and Bawa's method [10] for uniform mesh. For $\varepsilon = 1,\ p = 0$ (regular problem); our method reduces to the well-known Bickley scheme [16] for the regular problem. When $\mu^2 = -\lambda^2$ the method reduced to Aziz and Khan's method [15].

III. Convergence of the method

Convergence of the method introduced by the following theorem.

Theorem (1):

The method introduced by difference equation (14) for solving the boundary value problem (1) for $q(x) \ge 0$ gives a second- order convergent solution for arbitrary α,β with $\alpha+\beta = 1/2$ and a fourth-order convergent solution for $\alpha = 1/12,\ \beta = 5/12$.

Proof

The system (14) can be written in the matrix form

$$AY + h^2 R = C \tag{15}$$

where $A = (a_{i,i})$ is tridiagonal matrix of order $N-1$, with

$a_{i,i-1}$ is the coefficient of $y_{i-1},\ i = 2,3,...,N-1$,

$a_{i,i}$ is the coefficient of $y_i,\ i = 1,2,3,...,N-1$,

$a_{i,i+1}$ is the coefficient of $y_{i+1},\ i = 1,2,3,...,N-2$,

and $R = (\overline{r_1},\overline{r_2},\overline{r_3},...,\overline{r_{N-1}})^T$ where $\overline{r_i} = -h^2(\alpha r_{i-1} - 2\beta r_i + \alpha r_{i+1}), i = 1,2,...,N-1,$

and $C = (c_1,0,0,0,...,c_{N-1})^T$

where

$$c_1 = \alpha_0(\varepsilon - \frac{h}{2}\alpha p_2 - h\beta p_1 + \frac{3h\alpha}{2} p_0 - h^2\alpha q_0)$$

$$c_{N-1} = \alpha_1(\varepsilon - \frac{3h}{2}\alpha p_N - h^2\alpha q_N + h\beta p_{N-1} + \frac{h}{2}\alpha p_{N-2})$$

If we consider that

$\overline{Y} = (y(x_1), y(x_2), y(x_3),..., y(x_{N-1}))^T$

denotes to the exact solution vector, and

$T(h) = (T_1(h), T_2(h),,..., T_{N-1}(h))^T$

is the local truncation error vector then we have

$$A\overline{Y} + h^2 R = T(h) + C \tag{16}$$

Where $$T_i(h) = (-1+12\alpha)\frac{\varepsilon h^4}{12} y^{(4)}(\xi_i) + (-1+30\alpha)\frac{\varepsilon h^6}{360} y^{(6)}(\xi_i),\ x_{i-1} < \xi_i < x_{i+1} \tag{17}$$

for any choice of α and β whose sum is ½, except $\alpha = 1/12,\ \beta = 5/12$,

$$T_i(h) = (\frac{\varepsilon h^6}{240}) y^{(6)}(\xi_i), \quad x_{i-1} < \xi_i < x_{i+1} \tag{18}$$

From (15) and (16) we get

$$A(\overline{Y} - Y) = T(h) \tag{19}$$

$$A(E) = T(h) \tag{20}$$

where $E = \overline{Y} - Y = (e_1, e_2,..., e_N)^T$

From equation (14) we have

$$S_1 = \sum_{j=1}^{N-1} a_{1.j} = \varepsilon - 2\beta h^2 q_1 + \frac{3}{2}\alpha h p_0 - \frac{1}{2}\alpha h p_2 + \alpha h^2 q_2 - \beta h p_1,$$

$$S_2 = \sum_{j=1}^{N-1} a_{1.j} = h^2(\alpha q_{i-1} - 2\beta q_i + \alpha q_{i+1}) = h^2 \beta_i, i = 2,3,..., N-2,$$

$$S_{N-1} = \sum_{j=1}^{N-1} a_{1.j} = \varepsilon + \alpha h^2 q_{N-2} + \frac{1}{2}\alpha h p_{N-2} + \beta h p_N - \frac{3}{2}\alpha h p_{N-1} - 2\beta h^2 q_{N-1}$$

We can chose h sufficiently small so that the matrix A is irreducible and monotone [4].

It follows that A^{-1} exist and its elements are nonnegative. Hence from equation (20) we have

$$E = A^{-1}T(h) \tag{21}$$

and from the theory of matrices we have

$$\sum_{i=1}^{N-1} \bar{a}_{k,i} S_i = 1, \quad k = 1,2,...,N-1 \tag{22}$$

where $\bar{a}_{k,i}$ is the (k,i) element of the matrix A^{-1}. Therefore

$$\sum_{i=1}^{N-1} \bar{a}_{k,i} \le \frac{1}{\min_{1\le i \le N-1} S_i} = \frac{1}{h^2 B_m} \le \frac{1}{h^2 |B_m|}, \tag{23}$$

for some m between 1 and $N-1$
From equations (17), (21) and (22) we have

$$e_j = \sum_{i=1}^{N-1} \bar{a}_{j,i} T_i(h), \quad j = 1,2,....,N-1 \tag{24}$$

and therefore

$$|e_j| \le \frac{Kh^2}{|B_m|}, \quad j = 1,2,....,N-1 \tag{25}$$

where *K* is a constant, therefore

$$\|E\| = O(h^2) \tag{26}$$

which shows that the error is of order h^2 from which the convergence of the method is of second order.
When $\alpha = 1/12,\ \beta = 5/12$ equation (17) show that

$$|e_j| \le \frac{Kh^4}{|B_m|}, \quad j = 1,2,....,N-1 \tag{27}$$

Therefore

$$\|E\| = O(h^2) \tag{28}$$

Hence the method is of forth order convergence
It is worthy to mention here that the right form for equation (13) in (17)
$AY + h^2R = C$ also the right form for equation (14) is $A\bar{Y} + h^2R = T(h)C$ hence

$$c_1 = (\varepsilon - \lambda_1 h^2 q_0 + \tfrac{3}{2}\lambda_1 hp_0 + \lambda_2 hp_1 - \tfrac{\lambda_1}{2} hp_2)\alpha_0$$

$$c_{N-1} = (\varepsilon - \lambda_1 h^2 q_N + \tfrac{1}{2}\lambda_1 hp_{N-2} - \lambda_2 hp_{N-1} - \tfrac{3\lambda_1}{2} hp_N)\alpha_1$$

IV. Numerical example and discussion

We consider a numerical example t illustrates the performance of our present method and supports the theoretical analysis for second and fourth order convergence. All the computations were carried out using double precision arithmetic in order to keep the rounding errors negligible as compared to the discretization errors.

Example 1 (Doolan et al.[19])

$$\varepsilon y'' = y + \cos^2(\pi x) + 2\varepsilon\pi^2 \cos(2\pi x);$$

$$y(0) = y(1) = 0$$

The exact solution is given by

$$y(x) = \left[\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon}\right]/\left[1 + \exp(-1/\sqrt{\varepsilon}\right] - \cos^2(\pi x)$$

since $p(x) = 0$ and $q(x) = 1 > 0.$ q(x) = 1 > 0,
The approximate solution, Exact solution and the error at the nodal points with $\varepsilon = 1$ is tabulated in the following table:

x_i	y_i	Exact y_i	error
0.0	0	0	0
0.1	-0.149501	-0.149521	2.00819E-5
0.2	-0.0683609	-0.0682971	6.37847E-5
0.3	0.130901	0.131097	1.95818E-4
0.4	0.319183	0.319493	3.10347E-4
0.5	0.394771	0.395126	3.55134E-4
0.6	0.319183	0.319493	3.10347E-4

0.7	0.130901	0.131097	1.95818E-4
0.8	-0.0683609	-0.0682971	6.37847E-5
0.9	-0.149501	-0.149521	2.00819E-5
1.0	0	0	0

Maximum Absolute Error = 3.55134 E-4

We have described a numerical method for solving singular perturbation problems using parametric spline function. It is a practical method and can easily be implemented on a computer to solve such problem. The numerical calculations were carried out by mathematica program.

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